Postulates of Quantum Mechanics

1. **Wavefunction**

   The state of a system can be fully specified by its wavefunction in position space, which is a function of position \( \Psi(\mathbf{r}; t) \), or by its wavefunction in momentum space, \( \Psi(\mathbf{p}; t) \).

   The probability of finding the particle within a volume \( d\mathbf{r} \) around point \( \mathbf{r} \) is equal to \( | \Psi(\mathbf{r})|^2 \, d\mathbf{r} \). In order for the probability of finding the particle anywhere to be equal to unity, we require wavefunctions to be normalized:

   \[
   \int d\mathbf{r} |\Psi(\mathbf{r})|^2 = 1.
   \]

2. **Operators**

   States are transformed by linear operators:

   \[
   \hat{A}(\alpha \Psi_1 + \beta \Psi_2) = \alpha \hat{A} \Psi_1 + \beta \hat{A} \Psi_2.
   \]

   The commutator of two operators \( \hat{A}, \hat{B} \) is defined as

   \[
   [\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}.
   \]

   If the commutator of two operators is equal to zero we say that the operators commute. This is not always the case.

   To every physical observable corresponds a Hermitean operator. An operator \( \hat{A} \) is called Hermitean if

   \[
   \int d\mathbf{r} \Psi_1^*(\mathbf{r}) \hat{A} \Psi_2(\mathbf{r}) = \left \{ \int d\mathbf{r} \Psi_2^*(\mathbf{r}) \hat{A} \Psi_1(\mathbf{r}) \right \}^*
   \]

   for any two wavefunctions \( \Psi_1, \Psi_2 \).

3. **Uncertainty principle**

   The momentum operator is a vector with components \( \hat{p}_i = -i\hbar \frac{\partial}{\partial x^i} \), etc.
Therefore the operators for position and momentum satisfy the commutation relation
\[
\left[ \hat{x}, \hat{p}_x \right] = i\hbar.
\]
This means that it is not possible to determine simultaneously the position and momentum of a particle to precision better than Planck’s constant:
\[
\Delta p_x \Delta x \geq \frac{\hbar}{2}.
\]

4. Measurement

A given physical observable can take only certain values, which are the eigenvalues of the corresponding quantum mechanical operator.

When the wavefunction \( \Psi \) is an eigenfunction of the operator \( \hat{A} \) corresponding to the observable of interest, the determination of \( A \) always yields one result, the corresponding eigenvalue \( A_n \) of \( \hat{A} \). The state of the system changes to \( \Phi_n \) as a result of this measurement. When \( \Psi \) is not an eigenfunction of \( \hat{A} \), a single measurement of \( A \) yields a single result which is one of the eigenvalues of \( \hat{A} \); the probability that a particular eigenvalue \( A_n \) is measured is equal to \( |c_n|^2 \), where \( c_n \) is the coefficient of the eigenfunction \( \Phi_n \) in the expansion of the wavefunction \( \Psi \). Thus, the average of many measurements is given by the expectation value
\[
\langle A \rangle = \int d\mathbf{r} \Psi^*(\mathbf{r}) \hat{A} \Psi(\mathbf{r}).
\]

5. Time evolution

The time evolution of states satisfies the time-dependent Schrödinger equation,
\[
i\hbar \frac{\partial}{\partial t} \Psi(t) = \hat{H} \Psi(t)
\]
where \( \hat{H} \) is the Hamiltonian operator. Thus, usually (in Cartesian coordinates, assuming no magnetic fields are present) \( \hat{H} \) has the form
\[
\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\mathbf{r},t) = -\frac{\hbar^2}{2m} \nabla^2 + \hat{V}(\mathbf{r},t)
\]
where \( V \) is the potential energy operator (which often is independent of time).